Basics of algebra and calculus

June 6, 2025

The questions below are tied to basic mathematical concepts that are needed for the course in Economic Growth. Try first to answer these questions on your own. You can then consult the answers in the second part of the document to check your work. For those having problems with some topics, I recommend reviewing the following topics from *Sydsæter and Hammond, "Essential mathematics for economic analysis"* (SH) or *Larson and Edward, "Calculus"* (LE) which you can find in our library:

- Working with exponential functions (Chapter 4.9 (SH), 5.4 (LE))
- Working with logarithmic functions (Chapter 4.10 (SH), 5.1 (LE))
- Inverse functions (Chapter 5.3 (SH)+(LE))
- Differentiation (Chapter 6 (SH), 2 (LE))
- Optimization (Chapter 9 (SH), 3 (LE))
- Integration (Chapter 10.1-10.3 (SH), 4 (LE))

1. Exponents

Simplify the following expressions

(a) $(3x^2y^2)^2$

(b) x^0/x^{-2}

(c) $(3x^{-2})(2x^3)^2$

2. Logaritmic functions

Rewrite the following as functions of $\ln(x)$ and $\ln(y)$:

(a) $\ln(x^5 \frac{y}{3})$

(b) $\ln(x^{\frac{1}{3}} y^{-\frac{2}{3}})$

3. Derivatives

For each of the following functions, $y = f(x)$, provide the first and the second derivative.

(a) $y = f(x) = 4 + 2x^2$

(b) $y = f(x) = 2x^{\frac{1}{2}}$

(c) $y = f(x) = \frac{1}{2}\ln(2x + 1)$

4. Graphs of functions

For each of the following functions, $y = f(x)$, find the limits of the functions and provide a plot of the graph.

(a) $y = f(x) = x + x^{-1}$

(b) $y = f(x) = -4 - 2x + x^2$

(c) $y = f(x) = 2 + \ln x$

5. Inverse functions

For each of the following functions, $y = f(x)$, find the inverse function $f^{-1}(y)$.

(a) $y = f(x) = 3 + x^2$

(b) $y = f(x) = 5 - x$

(c) $y = f(x) = 3e^{2x}$

6. Solving equations

For each of the following equations, find the solution(s) or indicate whether no solution exists

(a) $-x + 2x^2 = -4$

(b) $2x + x^2 + 1 = 0$

(c) $\frac{1}{2} \ln x + 1 = 1$

7. Systems of equations

For each of the following systems of equations, find the solution(s) or indicate whether no solution exists

(a) $x + y = 5$, $2x - y = 1$

(b) $2x - 4y = 6$, $2y + x = 3$

(c) $x + 2y = 4$, $2y = 7 - x$

8. Partial derivatives

For each of the following functions, $z = f(x, y)$, provide first and the second derivatives with respect to x and with respect to y.

(a) $z = f(x, y) = x^\alpha y^\beta$

(b) $z = f(x, y) = \alpha \ln x + \beta \ln y$

(c) $z = f(x, y) = x + \alpha \ln(\beta y + 1)$

9. Global maxima/minima

For each of the following functions, $y = f(x)$, determine whether it attains a global maximum or minimum. If they do, solve for (x^*, y^*) at the critical points.

(a) $y = f(x) = 3 - 2x^2 + x$

(b) $y = f(x) = \frac{1}{2}x^2 - x + 1$

(c) $y = f(x) = -1 + 3x$

Answers

1. Exponents

(a) $(3x^2y)^3 = 27x^6y^3$

(b) $x^0/x^{-2} = x^2$

(c) $(3x)^{-2}(2x^3)^2 = \frac{4}{9}x^4$

2. Logaritmic functions

(a) $5\ln(x) + \frac{1}{3}\ln(y)$

(b) $\frac{1}{3}\ln(x) - \frac{2}{3}\ln(y)$

3. Derivatives

(a) $y = 4 + 2x^2$

$$y' = \frac{d}{dx}(4 + 2x^2) = 0 + 4x = 4x$$
$$y'' = \frac{d}{dx}(4x) = 4$$

(b) $y = 2x^{1/2}$

$$y' = 2 \cdot \frac{1}{2}x^{-1/2} = x^{-1/2} = \frac{1}{\sqrt{x}}$$
$$y'' = \frac{d}{dx}(x^{-1/2}) = -\frac{1}{2}x^{-3/2} = -\frac{1}{2x^{3/2}}$$

(c) $y = \frac{1}{2}\ln(2x + 1)$

$$y' = \frac{1}{2} \cdot \frac{1}{2x+1} \cdot 2 = \frac{1}{2x+1}$$
$$y'' = \frac{d}{dx}\left(\frac{1}{2x+1}\right) = -\frac{2}{(2x+1)^2}$$

4. Graphs of functions

Limits

(a)

$$x \mapsto -\infty, y \mapsto -\infty$$
$$x \mapsto \infty, y \mapsto \infty$$
$$x_- \mapsto 0, y \mapsto -\infty$$
$$x_+ \mapsto 0, y \mapsto \infty$$

(b)

$$x \mapsto -\infty, y \mapsto \infty$$
$$x \mapsto \infty, y \mapsto \infty$$

(c)

$$x_+ \mapsto 0, y \mapsto -\infty$$
$$x \mapsto \infty, y \mapsto \infty$$

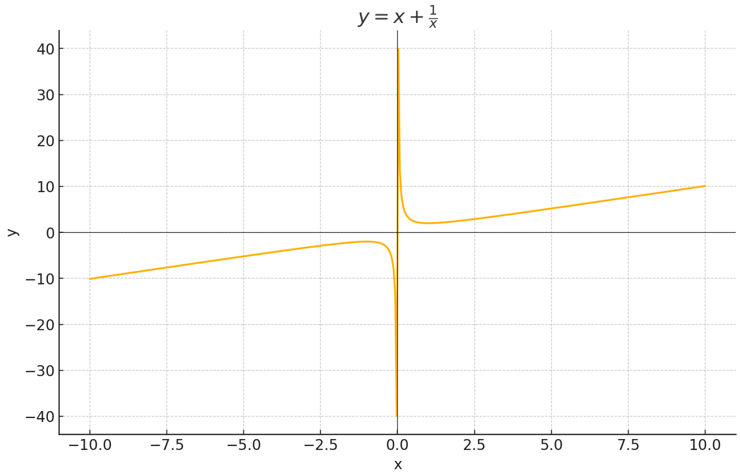

Figure 1: (a) $y = x + x^{-1}$

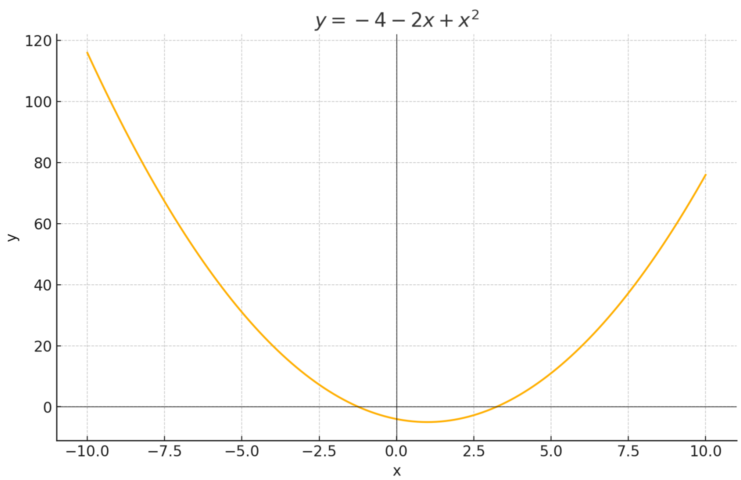

Figure 2: (b) $y = -4 - 2x + x^2$

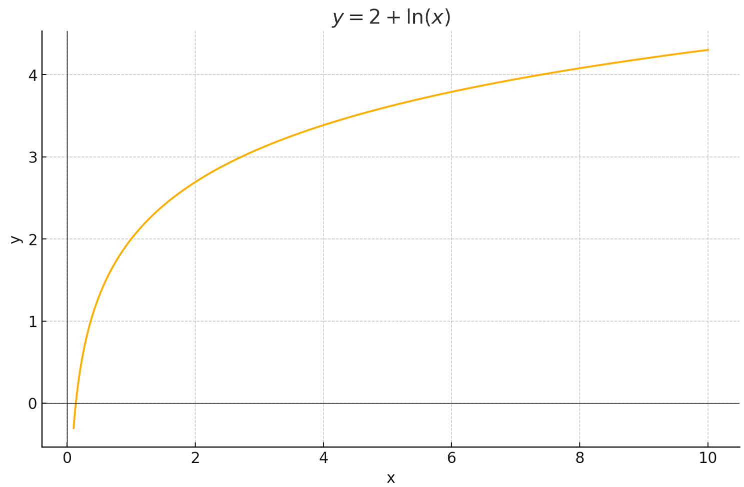

Figure 3: (c) $y = 2 + \ln x$

5. Inverse functions

(a) $y = 3 + x^2$

Restrict domain to $x \geq 0$:

$$\begin{aligned} y &= 3 + x^2 \\ y - 3 &= x^2 \\ x &= \sqrt{y-3} \\ f^{-1}(y) &= \sqrt{y-3} \end{aligned}$$

(b) $y = 5 - x$

$$\begin{aligned} y &= 5 - x \\ x &= 5 - y \\ f^{-1}(y) &= 5 - y \end{aligned}$$

(c) $y = 3e^{2x}$

$$\begin{aligned} y &= 3e^{2x} \\ \frac{y}{3} &= e^{2x} \\ \ln\left(\frac{y}{3}\right) = 2x \\ x &= \frac{1}{2}\ln\left(\frac{y}{3}\right) \\ f^{-1}(y) &= \frac{1}{2}\ln\left(\frac{y}{3}\right) \end{aligned}$$

6. Solving equations

(a) $-x + 2x^2 = -4$

$$2x^2 - x + 4 = 0$$

Use quadratic formula:

$$x = \frac{1 \pm \sqrt{(-1)^2 - 4 \cdot 2 \cdot 4}}{2 \cdot 2} = \frac{1 \pm \sqrt{-31}}{4} \Rightarrow \text{No real solution}$$

(b) $2x + x^2 + 1 = 0$

$$x^2 + 2x + 1 = 0$$
$$(x+1)^2 = 0 \Rightarrow x = -1$$

(c) $\frac{1}{2}\ln x + 1 = 1$

$$\frac{1}{2}\ln x = 0 \Rightarrow \ln x = 0 \Rightarrow x = 1$$

7. Systems of equations

(a)

$$\begin{aligned} x + y &= 5 \\ 2x - y &= 1 \\ \text{Add: } 3x = 6 &\Rightarrow x = 2 \\ y = 5 - 2 &= 3 \end{aligned}$$

(b)

$$\begin{aligned} 2x - 4y &= 6 \\ x - 2y &= 3 \quad \text{(Divide 1st by 2nd)} \end{aligned}$$
$$\text{Same equation} \Rightarrow \text{Infinite solutions}$$

(c)

$$\begin{aligned} x + 2y &= 4 \\ 2y &= 7 - x \Rightarrow x + 2y = x + 2y = 4 \text{ and } x + 2y = 7 \Rightarrow \text{No solution} \end{aligned}$$

8. Partial derivatives

(a) $z = f(x, y) = x^{\alpha} y^{\beta}$

First Partial Derivatives

Partial with respect to x: Use the power rule while treating y^{β} as constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}\left(x^{\alpha} y^{\beta}\right) = y^{\beta} \cdot \frac{d}{dx}(x^{\alpha}) = y^{\beta} \cdot \alpha x^{\alpha - 1}$$

Partial with respect to y: Similarly, treat x^{α} as constant and differentiate y^{β}:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}\left(x^{\alpha} y^{\beta}\right) = x^{\alpha} \cdot \frac{d}{dy}(y^{\beta}) = x^{\alpha} \cdot \beta y^{\beta - 1}$$

Second Partial Derivatives

Second with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\alpha x^{\alpha - 1} y^{\beta}\right) = y^{\beta} \cdot \alpha(\alpha - 1) x^{\alpha - 2}$$

Mixed partial:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\beta x^{\alpha} y^{\beta - 1}\right) = \alpha\beta x^{\alpha - 1} y^{\beta - 1} \tag{1}$$

Second with respect to y:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\beta x^{\alpha} y^{\beta - 1}\right) = x^{\alpha} \cdot \beta(\beta - 1) y^{\beta - 2}$$

(b) $z = f(x, y) = \alpha \ln x + \beta \ln y$

First Partial Derivatives

Partial with respect to x: Use the derivative of $\ln x$:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(\alpha \ln x + \beta \ln y) = \alpha \cdot \frac{1}{x}$$

Partial with respect to y: Use the derivative of $\ln y$, treating $\ln x$ as constant:

$$\frac{\partial z}{\partial y} = \frac{\partial}{\partial y}(\alpha \ln x + \beta \ln y) = \beta \cdot \frac{1}{y}$$

Second Partial Derivatives

Second with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\alpha \cdot \frac{1}{x}\right) = -\alpha \cdot \frac{1}{x^2}$$

Mixed partial:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\beta}{y}\right) = 0 \tag{2}$$

Second with respect to y:

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\beta \cdot \frac{1}{y}\right) = -\beta \cdot \frac{1}{y^2}$$

(c) $z = f(x, y) = x + \alpha \ln(\beta y + 1)$

First Partial Derivatives

Partial with respect to x: Only the term x depends on x, the rest is treated as constant:

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x}(x + \alpha \ln(\beta y + 1)) = 1$$

Partial with respect to y: Apply the chain rule to differentiate $\ln(\beta y + 1)$:

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{\partial}{\partial y}(\alpha \ln(\beta y + 1)) = \alpha \cdot \frac{1}{\beta y + 1} \cdot \frac{d}{dy}(\beta y + 1) \\ &= \alpha \cdot \frac{1}{\beta y + 1} \cdot \beta = \frac{\alpha\beta}{\beta y + 1}\end{aligned}$$

Second Partial Derivatives

Second with respect to x:

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial x}(1) = 0$$

Mixed partial:

$$\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial}{\partial x}\left(\frac{\alpha\beta}{\beta y + 1}\right) = 0 \tag{3}$$

Second with respect to y: Apply the quotient rule or differentiate the result using the chain rule:

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2} &= \frac{d}{dy}\left(\frac{\alpha\beta}{\beta y + 1}\right) = \alpha\beta \cdot \frac{d}{dy}\left(\frac{1}{\beta y + 1}\right) \\ &= \alpha\beta \cdot \left(-\frac{\beta}{(\beta y + 1)^2}\right) \\ &= \frac{-\alpha\beta^2}{(\beta y + 1)^2}\end{aligned}$$

9. Global maxima/minima

(a) $y = 3 - 2x^2 + x$

$$y' = -4x + 1 = 0 \Rightarrow x = \frac{1}{4}$$
$$y = 3 - 2\left(\frac{1}{4}\right)^2 + \frac{1}{4} = 3 - \frac{1}{8} + \frac{1}{4} = \frac{27}{8}$$
$$\text{Max at } (\frac{1}{4}, \frac{27}{8})$$

(b) $y = \frac{1}{2}x^2 - x + 1$

$$y' = x - 1 = 0 \Rightarrow x = 1$$
$$y = \frac{1}{2}(1)^2 - 1 + 1 = \frac{1}{2}$$
$$\text{Min at } (1, \frac{1}{2})$$

(c) $y = -1 + 3x$

Linear function: no max or min unless domain is restricted.